CÁLCULO INTEGRAL

Definição de Integração

Integração: operação inversa da derivação. O cálculo integral tem por objetivo achar uma função F(x) tal que a derivada F'(x) seja igual a uma função dada f(x).

$$F'(x) = \frac{dF(x)}{dx} = f(x)$$

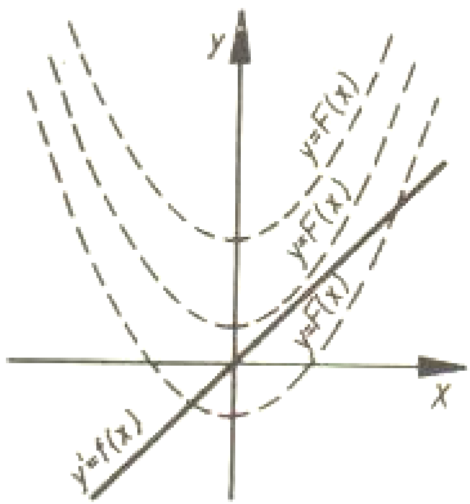
donde, por integração

"A integral indefinida"

$$\int f(x)dx = F(x) + C$$

A constante de integração C desaparece no momento da derivação, pois a derivada de uma constante é nula.

Interpretação geométrica da integral definida



Como a figura indica, há uma infinidade de curvas y = F(x) de inclinação y' = f(x). Todas as curvas são idênticas mas deslocadas paralelamente ao eixo y. A todo valor de C corresponde uma só curva. Se a curva passa por um ponto (x_o, y_o), encontra-se:

$$C = y_o - F(x_o)$$

“A integral definida”
A forma geral da integral definida é:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Diz-se então que se integra entre os limites a e b: introduzir x = b, depois x = a em F(x) e subtrai o segundo valor do primeiro.

Integração – Regras Gerais

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad se\ n \neq -1$$

$$\int \frac{dx}{x} = \ln x + C$$

$$\int [u(x) \pm v(x)]\, dx = \int u(x)\, dx \pm \int v(x)\, dx$$

$$\int \frac{u'(x)}{u(x)}\, dx = \ln u(x) + C$$

$$\int u(x).u'(x)\, dx = \frac{1}{2}[u(x)]^2 + C$$

Integração por Partes

$$\int u(x).v'(x)dx = u(x).v(x) - \int u'(x).v(x)dx$$

Método de substituição

$$\int f(x)dx = \int f[\varphi(z)]\,.\,\varphi(z)\, dz$$

onde $x = \varphi(z)$ e $dx = \varphi'(z)\, dz$

Exemplo:

$$F(x) = \sqrt{3x-5}\, dx$$

Façamos $3x - 5 = z$, onde $z' = \frac{dz}{dx} = 3$

Obtém-se $dx = \frac{dz}{3}$, integral em função de z:

$$F(x) = \frac{1}{3}\int \sqrt{z}\ dz = \frac{2}{9} z\sqrt{z} + C$$

Substituindo z por seu valor

$$F(x) = \frac{2}{9}(3x-5)\sqrt{3x-5} + C$$

Integrais Fundamentais (sem a constante de integração C)

$$\int e^x dx = e^x \qquad \int \ln x\ dx = x\ln x - x$$

$$\int a^x dx = \frac{a^x}{\ln a} \qquad \int \frac{dx}{x-a} = \ln(x-a)$$

$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{a-b}\ln\frac{x-a}{x-b} \qquad (a \neq b)$$

$$\int \frac{dx}{(x-a)^n} = -\frac{1}{(n-1)(x-a)^{n-1}} \qquad (n \neq 1)$$

$$\int \frac{dx}{x^2 - a^2} = -\frac{1}{a} ar\coth\frac{x}{a} = \frac{1}{2a}\ln\frac{x-a}{x+a} \qquad (x > a)$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a}\arctan\frac{x}{a} = \frac{1}{2a}\ln\frac{a+x}{a-x} \qquad (x < a)$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} \qquad \int \frac{xdx}{x^2 + a^2} = \frac{1}{2}\ln(x^2 + a^2)$$

$$\int \frac{dx}{(x^2 + a^2)^2} = \frac{x}{2a^2(x^2 + a^2)} + \frac{1}{2a^3}\arctan\frac{x}{a}$$

$$\int \frac{dx}{(x^2 + a^2)^n} = \frac{x}{2a^2(n-1)(x^2 + a^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)}\int \frac{dx}{(x^2 + a^2)^{n-1}}$$

$$\int \sqrt{x}dx = \frac{2}{3}\sqrt{x^3} \qquad \int \frac{dx}{\sqrt{x}} = 2\sqrt{x}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = arcsen\frac{x}{a} \qquad \int \frac{dx}{\sqrt{ax+b}} = \frac{2}{a}\sqrt{ax+b}$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \operatorname{arccos}h\frac{x}{a} = \ln\,(x + \sqrt{x^2 - a^2}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \operatorname{arccos}h\frac{x}{a} = \ln\,(x + \sqrt{x^2 + a^2}$$

$$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\,arcsen\frac{x}{a}$$

$$\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\operatorname{arccos}h\frac{x}{a}$$

$$\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\,arcsenh\frac{x}{a}$$

$$\int sen\ x\ dx = -\cos x$$

$$\int sen^2 x\ dx = \frac{x}{2} - \frac{1}{4}sen\ (2x)$$

$$\int sen^3 x\ dx = -\frac{3}{4}\cos\ x + \frac{1}{12}\cos\ (3x)$$

$$\int sen^n x\ dx = -\frac{1}{n}\cos\ x\ .\ sen^{n-1}x + \frac{n-1}{n}\int sen^{n-2} x\ dx$$

$$\int sen\ (ax)\ dx = -\frac{1}{a}\cos\ (ax)$$

$$\int \cos\ x\ dx = sen\ x$$

$$\int \cos^2\ x\ dx = \frac{x}{2} + \frac{1}{4}sen\ (2x)$$

$$\int \cos^3\ x\ dx = \frac{3}{4}sen\ x + \frac{1}{12}sen\ (3x)$$

$$\int \cos^n\ x\ dx = \frac{1}{n}sen\ x\ .\ \cos^{n-1}\ x + \frac{n-1}{n}\int \cos^{n-2} x\ dx$$

$$\int \cos\ (ax)\ dx = \frac{1}{a}sen\ (ax)$$

$$\int \tan x\, dx = -\ln \cos x \qquad \int \tan (ax)\, dx = -\frac{1}{a}\ln \cos (ax)$$

$$\int \tan^2 x\, dx = \tan x - x$$

$$\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx \qquad (n \neq 1)$$

$$\int \cot x\, dx = \ln sen\, x \qquad \int \cot (ax)\, dx = \frac{1}{a}\ln sen\, (ax)$$

$$\int \cot^2 x\, dx = x - \cot x$$

$$\int \cot^n x\, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx \qquad (n \neq 1)$$

$$\int \frac{dx}{sen\, x} = \ln \tan \frac{x}{2} \qquad \int \frac{dx}{sen^2\, x} = -\cot x$$

$$\int \frac{dx}{sen^n\, x} = -\frac{1}{n-1} \cdot \frac{\cos x}{sen^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{sen^{n-2} x} \qquad (n \neq 1)$$

$$\int \frac{dx}{\cos x} = \ln \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \qquad \int \frac{dx}{\cos^2 x} = \tan x$$

$$\int \frac{dx}{\cos^n x} = \frac{1}{n-1} \cdot \frac{sen\, x}{\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} x} \qquad (n \neq 1)$$

$$\int \frac{dx}{1 + sen\, x} = \tan\left(\frac{x}{2} - \frac{\pi}{4}\right) \qquad \int \frac{dx}{1 - sen\, x} = -\cot\left(\frac{x}{2} - \frac{\pi}{4}\right)$$

$$\int \frac{dx}{1 + \cos x} = \tan \frac{x}{2} \qquad \int \frac{dx}{1 - \cos x} = -\cot \frac{x}{2}$$

$$\int sen\, (ax)\, sen\, (bx)\, dx = -\frac{sen\, (ax + bx)}{2(a+b)} + \frac{sen\, (ax - bx)}{2(a-b)} \qquad (\, |a| \neq |b|\,)$$

$$\int sen\, (ax) \cos (bx)\, dx = -\frac{\cos (ax + bx)}{2(a+b)} - \frac{\cos (ax - bx)}{2(a-b)} \qquad (\, |a| \neq |b|\,)$$

$$\int \cos (ax) \cos (bx)\, dx = \frac{sen\, (ax + bx)}{2(a+b)} + \frac{sen\, (ax - bx)}{2(a-b)} \qquad (\, |a| \neq |b|\,)$$

$$\int x^n sen\,(ax)\,dx = -\frac{x^n}{a}\cos\,(ax) + \frac{n}{a}\int x^{n-1}\cos\,(ax)\,dx$$

$$\int x^n \cos\,(ax)\,dx = \frac{x^n}{a} sen\,(ax) - \frac{n}{a}\int x^{n-1} sen\,(ax)\,dx$$

$$\int arcsen\;x\,dx = x\,.\,arcsen\;x + \sqrt{1-x^2}$$

$$\int \arccos\;x\,dx = x\,.\,\arccos\;x - \sqrt{1-x^2}$$

$$\int \arctan\;x\,dx = x\,.\,\arctan\;x - \frac{1}{2}\ln(1+x^2)$$

$$\int arc\cot\;x\,dx = x\,.\,arc\cot\;x + \frac{1}{2}\ln(1+x^2)$$

$$\int senh\;x\,dx = \cosh\;x$$

$$\int senh^2\;x\,dx = \frac{1}{4} senh\,(2x) - \frac{x}{2}$$

$$\int senh^n\;x\,dx = \frac{1}{n}\cosh\;x\,.\,senh^{n-1}x - \frac{n-1}{n}\int senh^{n-2}x\,dx$$

$$\int senh\,(ax)\,dx = \frac{1}{a}\cosh\,(ax)$$

$$\int \cosh\;x\,dx = senh\;x$$

$$\int \cosh^2\;x\,dx = \frac{1}{4} senh\,(2x) + \frac{x}{2}$$

$$\int \cosh^n\;x\,dx = \frac{1}{n} senh\;x\,.\,\cosh^{n-1}\;x + \frac{n-1}{n}\int \cos^{n-2}\;x\,dx$$

$$\int \cosh\,(ax)\,dx = \frac{1}{a} senh\,(ax)$$

$$\int \tanh\;x\,dx = \ln\cosh\;x$$

$$\int \tanh^2\;x\,dx = x - \tanh\;x$$

$$\int \tanh^n x\, dx = -\frac{1}{n-1}\tanh^{n-1} x + \int \tanh^{n-2} x\, dx \qquad (n \neq 1)$$

$$\int \tanh\,(ax)\, dx = \frac{1}{a}\ln \cosh\,(ax)$$

$$\int \coth x\, dx = \ln senh\, x$$

$$\int \coth^2 x\, dx = x - \coth x$$

$$\int \coth^n x\, dx = -\frac{1}{n-1}\coth^{n-1} x + \int \coth^{n-2} x\, dx \qquad (n \neq 1)$$

$$\int \coth\,(ax)\, dx = \frac{1}{a}\ln senh\,(ax)$$

$$\int \frac{dx}{senh\, x} = \ln \tanh \frac{x}{2}$$

$$\int \frac{dx}{senh^2\, x} = -\coth x$$

$$\int \frac{dx}{\cosh x} = 2\arctan e^x$$

$$\int \frac{dx}{\cosh^2 x} = \tanh x$$

$$\int arcsenh\, x\, dx = x\, .\, arcsenh\, x - \sqrt{x^2 + 1}$$

$$\int \operatorname{arccos}h\, x\, dx = x\, .\, \operatorname{arccos}h\, x - \sqrt{x^2 - 1}$$

$$\int \arctan h\, x\, dx = x\, .\, \arctan h\, x + \frac{1}{2}\ln\left(1 - x^2\right)$$

$$\int arc\coth x\, dx = x\, .\, arc\coth x + \frac{1}{2}\ln\left(x^2 - 1\right)$$

$$\int sen^m x\, .\cos^n x\, dx = \frac{1}{m+n} sen^{m+1} x\, .\cos^{n-1} x + \frac{n-1}{m+n}\int sen^m x\, .\cos^{n-2} x\, dx$$

Para n ímpar, a integral restante vale:

$$\int sen^m x \,.\, \cos x \; dx = \frac{sen^{m+1} x}{m+1}$$

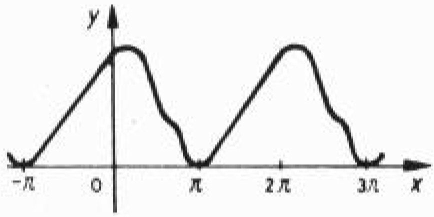
Aritmética

Séries de Fourier

D 12

Séries de Fourier

Generalidades: Toda função periódica $f(x)$, que pode ser decomposta num intervalo de periodicidade $-\pi \geqslant x \geqslant \pi$ em um número finito de partes inteiras, neste intervalo, pode ser decomposta em séries convergentes da seguinte forma ($x = \omega t$):



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

Calcula-se cada um de seus coeficientes segundo:

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\,dx \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\,dx$$

para cada $k = 0, 1, 2, \ldots$

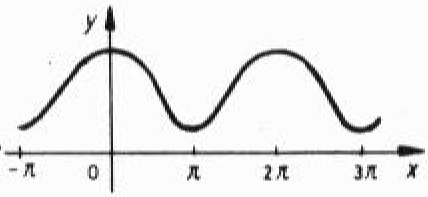
Simplificação do Cálculo dos Coeficientes por Simetria:

Função para: $f(x) = f(-x)$

$$a_k = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos(kx)\,dx$$

para $k = 0, 1, 2, \ldots$

$$b_k = 0$$

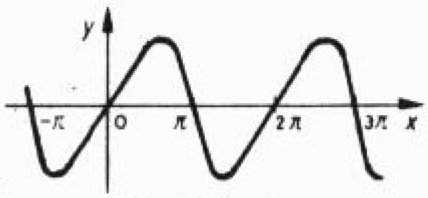


Função ímpar: $f(x) = -f(-x)$

$$a_k = 0$$

$$b_k = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin(kx)\,dx$$

para $k = 0, 1, 2, \ldots$



Simetria Par	Simetria ímpar
No caso de $f(x) = f(-x)$ e $f(\frac{\pi}{2} + x) = -f(\frac{\pi}{2} - x)$ temos:	No caso de $f(x) = -f(-x)$ e $f(\frac{\pi}{2} + x) = -f(\frac{\pi}{2} - x)$ temos:
$a_k = \frac{4}{\pi} \int_{0}^{\pi/2} f(x) \cos(kx)\,dx$ para $k = 1, 3, 5, \ldots$	$b_k = \frac{4}{\pi} \int_{0}^{\pi/2} f(x) \sin(kx)\,dx$ para $k = 1, 3, 5, \ldots$
$a_k = 0$ para $k = 0, 2, 4, \ldots$	$a_k = 0$ para $k = 0, 1, 2, \ldots$
$b_k = 0$ para $k = 1, 2, 3, \ldots$	$b_k = 0$ para $k = 2, 4, 6, \ldots$

Aritmética

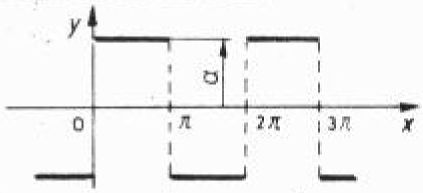
Séries de Fourier

D 13

Tabela de Desenvolvimento em Séries de Fourier

$y = a$ para $0 < x < \pi$

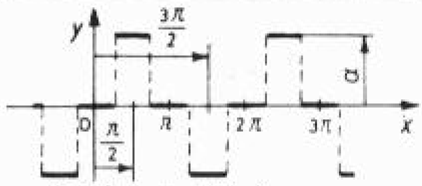
$y = -a$ para $\pi < x < 2\pi$



$$y = \frac{4a}{\pi}\left[\sin x + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \dots\right]$$

$y = a$ para $a < x < \pi - a$

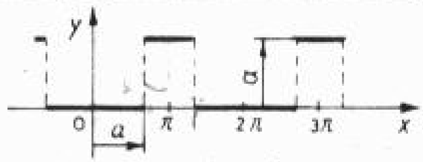
$y = -a$ para $\pi + a < x < 2\pi - a$



$$y = \frac{4a}{\pi}\left[\cos a \sin x + \frac{1}{3}\cos(3a)\sin(3x) + \frac{1}{5}\cos(5a)\sin(5x) + \dots\right]$$

$y = a$ para $a < x < 2\pi - a$

$y = f(2\pi + x)$

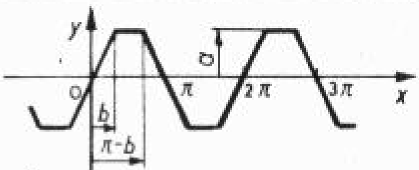


$$y = \frac{2a}{\pi}\left[\frac{\pi - a}{2} + \frac{\sin(\pi - a)}{1}\cos x + \frac{\sin 2(\pi - a)}{2}\cos(2x) + \frac{\sin 3(\pi - a)}{3}\cos(3x) + \dots\right]$$

$y = ax/b$ para $0 \leq x \leq b$

$y = a$ para $b \leq x \leq \pi - b$

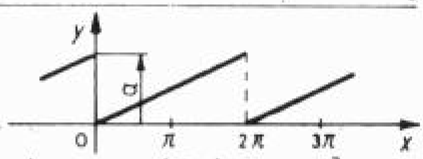
$y = a(\pi - x)/b$ para $\pi - b \leq x \leq \pi$



$$y = \frac{4}{\pi}\frac{a}{b}\left[\frac{1}{1^2}\sin b \sin x + \frac{1}{3^2}\sin(3b)\sin(3x) + \frac{1}{5^2}\sin(5b)\sin(5x) + \dots\right]$$

$y = \dfrac{ax}{2\pi}$ para $0 < x < 2\pi$

$y = f(2\pi + x)$



$$y = \frac{a}{2} - \frac{a}{\pi}\left[\frac{\sin x}{1} + \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} + \dots\right]$$

Aritmética

Séries de Fourier

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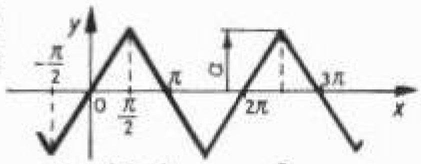
Cont. de D 13

$y = 2ax/\pi$ para $0 \leq x \leq \pi/2$

$y = 2a(\pi - x)/\pi$ para $\pi/2 \leq x \leq \pi$

$y = -f(\pi + x)$

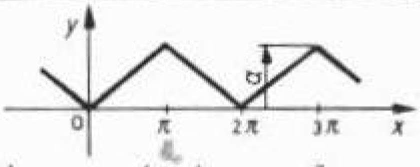


$$y = \frac{8}{\pi^2} a \left[\sin x - \frac{\sin(3x)}{3^2} + \frac{\sin(5x)}{5^2} - \ldots \right]$$

$y = ax/\pi$ para $0 \leq x \leq \pi$

$y = a(2\pi - x)/\pi$ para $\pi \leq x \leq 2\pi$

$y = f(2\pi + x)$

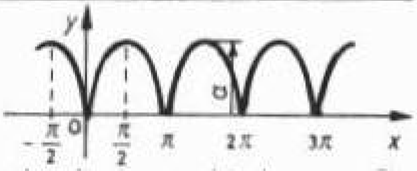


$$y = \frac{a}{2} - \frac{4a}{\pi^2} \left[\frac{\cos x}{1^2} + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + \ldots \right]$$

$y = a \sin x$ para $0 \leq x \leq \pi$

$y = -a \sin x$ para $\pi \leq x \leq 2\pi$

$y = f(\pi + x)$

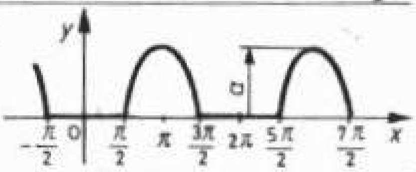


$$y = \frac{2a}{\pi} - \frac{4a}{\pi} \left[\frac{\cos(2x)}{1 \cdot 3} + \frac{\cos(4x)}{3 \cdot 5} + \frac{\cos(6x)}{5 \cdot 7} + \ldots \right]$$

$y = 0$ para $0 \leq x \leq \pi/2$

$y = a \sin\left(x - \frac{\pi}{2}\right)$ para $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$

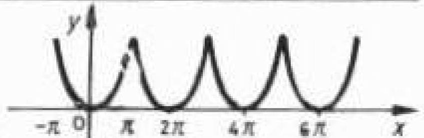
$y = f(2\pi + x)$



$$y = \frac{2a}{\pi} \left[\frac{1}{2} + \frac{\pi}{4} \cos x + \frac{\cos(2x)}{2^2 - 1} - \frac{\cos(4x)}{4^2 - 1} + \frac{\cos(6x)}{6^2 - 1} - \ldots \right]$$

$y = x^2$ para $-\pi \leq x \leq \pi$

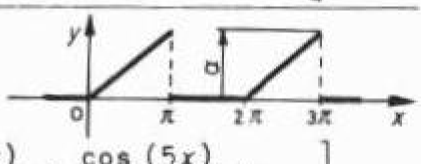
$y = f(-x) = f(2\pi + x)$



$$y = \frac{\pi^2}{3} - 4 \left[\frac{\cos x}{1^2} + \frac{\cos(2x)}{2^2} + \frac{\cos(3x)}{3^2} + \ldots \right]$$

$y = ax/\pi$ para $0 \leq x \leq \pi$

$y = f(2\pi + x)$



$$y = \frac{a}{2} - \frac{2a}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + \ldots \right] + \frac{a}{2} \left[\frac{\sin x}{1} + \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} + \ldots \right]$$

Aritmética

Números complexos

Números complexos

Generalidades:

$z = r\,e^{i\varphi}$

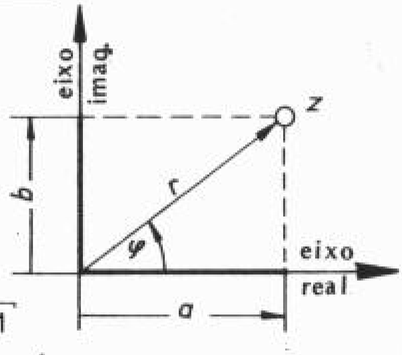
a = parte real de z

b = parte imaginária de z

r = módulo de z

φ = argumento de z

a e b são reais



$$i = \sqrt{-1}$$

$i^1 = +i$	$i^{-1} = -i$
$i^2 = -1$	$i^{-2} = -1$
$i^3 = -i$	$i^{-3} = +i$
$i^4 = +1$	$i^{-4} = +1$
$i^5 = +i$	$i^{-5} = -i$

etc

Observação: Em eletrotécnica, substitui-se i por j para evitar confusões.

Em um sistema de coordenadas cartesianas:

$$z = a + ib$$

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

$$z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$$

$$z_1 \quad z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

$$\frac{z_1}{z_2} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\frac{-a_1 b_2 + a_2 b_1}{a_2^2 + b_2^2}$$

$$a^2 + b^2 = (a + ib)(a - ib)$$

$$\sqrt{a \pm ib} = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \pm i\sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}}$$

Se $a_1 = a_2$ e $b_1 = b_2$, então $z_1 = z_2$

Aritmética

Números complexos

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Números complexos
(continuação)

Em um sistema de coordenadas polares:

$$z = r(\cos\varphi + i \cdot \text{sen}\,\varphi) = a + ib$$

$$r = +\sqrt{a^2 + b^2}$$

$$\varphi = \arctan\frac{b}{a}$$

$$\text{sen}\,\varphi = \frac{b}{r} \quad \Big| \quad \cos\varphi = \frac{a}{r} \quad \Big| \quad \tan\varphi = \frac{b}{a}$$

$$z_1 \cdot z_2 = r_1 \cdot r_2 \left[\cos(\varphi_1 + \varphi_2) + i \cdot \text{sen}\,(\varphi_1 + \varphi_2)\right]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot \left[\cos(\varphi_1 - \varphi_2) + i \cdot \text{sen}\,(\varphi_1 - \varphi_2)\right] \quad (z_2 \neq 0)$$

$$z^n = r^n\left[\cos(n\varphi) + i \cdot \text{sen}\,(n\varphi)\right] \quad (n > 0,\ \text{inteiro})$$

$$\sqrt[n]{z} = \left|\sqrt[n]{r}\right| \left(\cos\frac{\varphi + 2\pi k}{n} + i \cdot \text{sen}\,\frac{\varphi + 2\pi k}{n}\right)$$

$$\sqrt[n]{1} = \cos\frac{2\pi k}{n} + i \cdot \text{sen}\,\frac{2\pi k}{n} \quad (\text{enésima raiz inteira})$$

nas fórmulas d 176 e d 177: $k = 0, 1, 2, \ldots, n-1$

$$e^{i\widehat{\varphi}} = \cos\varphi + i \cdot \text{sen}\,\varphi$$

$$e^{-i\widehat{\varphi}} = \cos\varphi - i \cdot \text{sen}\,\varphi = \frac{1}{\cos\varphi + i \cdot \text{sen}\,\varphi}$$

$$\left|e^{-i\widehat{\varphi}}\right| = \sqrt{\cos^2\varphi + \text{sen}^2\varphi} = 1$$

$$\cos\varphi = \frac{e^{i\widehat{\varphi}} + e^{-i\widehat{\varphi}}}{2} \quad \Big| \quad \text{sen}\,\varphi = \frac{e^{i\widehat{\varphi}} - e^{-i\widehat{\varphi}}}{2i}$$

$$\ln z = \ln r + i(\widehat{\varphi} + 2\pi k) \quad (k = 0, \pm 1, \pm 2, \ldots)$$

Se $r_1 = r_2$ e $\widehat{\varphi}_1 = \widehat{\varphi}_2 + 2\pi k$, então $z_1 = z_2$

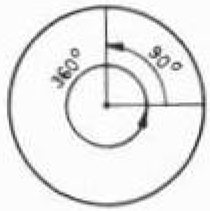
Funções circulares
Noções fundamentais

E 1

Grau e radiano de um ângulo plano

Representação detalhada

Um ângulo plano é expresso seja em graus α, seja em radianos $\widehat{\alpha}$. A relação seguinte existe entre estas duas graduações:



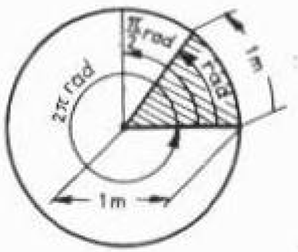
$$\widehat{\alpha} = \frac{\pi \text{ rad}}{180^0} \cdot \alpha = \frac{\text{rad}}{57{,}2958^0} \alpha$$

Unidades de α: grau (°); minuto (') e segundo ('').

Unidades de $\widehat{\alpha}$: rad; –; m/m

1 radiano (rad) é o ângulo no centro de um círculo de 1 m de raio se o arco interceptado vale também 1 m.

Donde:



$$1 \text{ rad} = \frac{1 \text{ m}}{1 \text{ m}}$$

Assim, o radiano se exprime por um número puro, como no quadro abaixo: a notação rad pode ser suprimida.

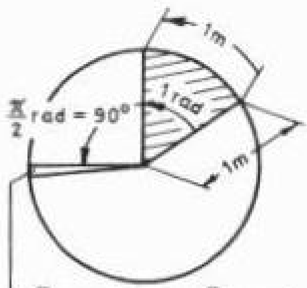
α	0°	30°	45°	60°	75°	90°	180°	270°	360°
$\widehat{\alpha}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5}{12}\pi$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
	0	0,52	0,79	1,05	1,31	1,57	3,14	4,71	6,28

Representação simplificada usual
(utilizada neste formulário)

Se $\alpha = 1°$ então $\widehat{\alpha} = \frac{\pi}{180}$ rad

Donde $1° = \frac{\pi}{180}$ rad

Então, os dois ângulos são iguais ($\alpha = \widehat{\alpha}$)



$\frac{\pi}{2}$ rad : 90 = $\frac{\pi}{180}$ rad
= 1 Grado = 1°

$\widehat{\alpha} = 1 \text{ rad} = 57{,}2958°$

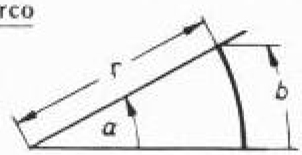
Unidades: 1°; – ; rad; m/m

Funções circulares

Noções gerais

E 2

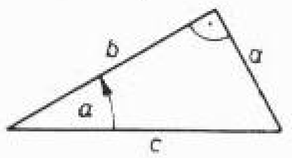
Comprimento de arco



O comprimento do arco b de um círculo de raio r e ângulo $\widehat{\alpha}$ no centro vale:

$$b = r\widehat{\alpha}$$

O triângulo retângulo



$$\text{sen } \alpha = \frac{\text{lado oposto}}{\text{hipotenusa}} = \frac{a}{c}$$

$$\cos \alpha = \frac{\text{cateto adjacente}}{\text{hipotenusa}} = \frac{b}{c}$$

$$\tan \alpha = \frac{\text{cateto oposto}}{\text{cateto adjacente}} = \frac{a}{b} \qquad \cot \alpha = \frac{\text{cateto adjacente}}{\text{cateto oposto}} = \frac{b}{a}$$

Valores das funções dos ângulos importantes

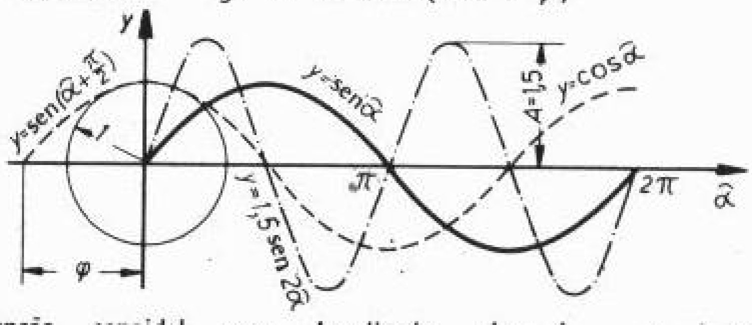
ângulo α	0°	30°	45°	60°	75°	90°	180°	270°	360°
sen α	0	0,500	0,707	0,866	0,966	1	0	−1	0
cos α	1	0,866	0,707	0,500	0,259	0	−1	0	1
tan α	0	0,577	1,000	1,732	3,732	∞	0	∞	0
cot α	∞	1,732	1,000	0,577	0,268	0	∞	0	∞

Relações entre as funções senoidais e cossenoidais

Equações fundamentais

Função senoidal $y = A \text{ sen } (k\alpha - \varphi)$

Função co-senoidal $y = A \cos (k\alpha - \varphi)$



——— Função senoidal com Amplitude $A = 1$ e $k = 1$

—·— Função senoidal com Amplitude $A = 1,5$ e $k = 2$

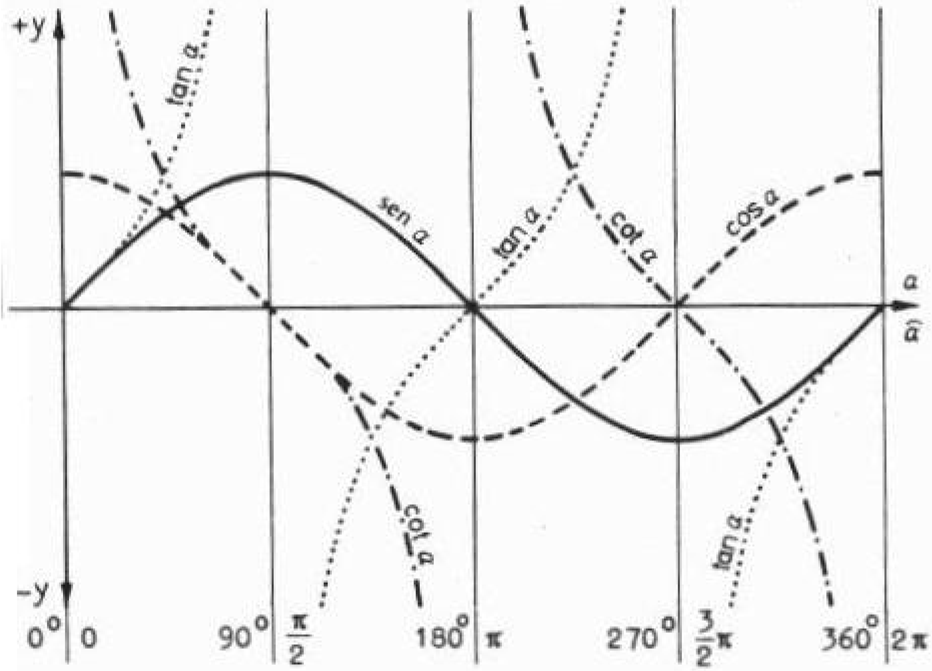
- - - - Função co-senoidal com Amplitude $A = 1$ e $k = 1$ ou função seno com defasagem $\varphi = -\frac{\pi}{2}$

Funções circulares

Os quadrantes

E 3

sen (90° − α)	=	+ cos α	sen (90° + α)	=	+ cos α
cos(")	=	+ sen α	cos(")	=	− sen α
tan(")	=	+ cot α	tan(")	=	− cot α
cot(")	=	+ tan α	cot(")	=	− tan α
sen (180° − α)	=	+ sen α	sen (180° + α)	=	− sen α
cos(")	=	− cos α	cos(")	=	− cos α
tan(")	=	− tan α	tan(")	=	+ tan α
cot(")	=	− cot α	cot(")	=	+ cot α
sen (270° − α)	=	− cos α	sen (270° + α)	=	− cos α
cos(")	=	− sen α	cos(")	=	+ sen α
tan(")	=	+ cot α	tan(")	=	− cot α
cot(")	=	+ tan α	cot(")	=	− tan α
sen (360° α)	=	− sen α	sen (360° + α)	=	+ sen α
cos(")	=	+ cos α	cos(")	=	+ cos α
tan(")	=	− tan α	tan(")	=	+ tan α
cot(")	=	− cot α	cot(")	=	+ cot α
sen (− α)	=	− sen α	sen (α ± n 360°)	=	+ sen α
cos(")	=	+ cos α	cos(")	=	+ cos α
tan(")	=	− tan α	tan(α ± n 180°)	=	+ tan α
cot(")	=	− cot α	cot(")	=	+ cot α



Funções circulares

Relações goniométricas

E 4

Relações fundamentais

$$\text{sen}^2\alpha + \cos^2\alpha = 1 \qquad \tan\alpha \cdot \cot\alpha = 1$$

$$1 + \tan^2\alpha = \frac{1}{\cos^2\alpha} \qquad 1 + \cot^2\alpha = \frac{1}{\text{sen}^2\alpha}$$

Funções de uma soma ou diferença de ângulos

$$\text{sen}(\alpha \pm \beta) = \text{sen}\,\alpha \cdot \cos\beta \pm \cos\alpha \cdot \text{sen}\,\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha \cdot \cos\beta \mp \text{sen}\,\alpha \cdot \text{sen}\,\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \cdot \tan\beta}$$

$$\cot(\alpha \pm \beta) = \frac{\cot\alpha \cdot \cot\beta \mp 1}{\cot\beta \pm \cot\alpha}$$

Soma e diferença de funções trigonométricas

$$\text{sen}\,\alpha + \text{sen}\,\beta = 2\,\text{sen}\,\frac{\alpha+\beta}{2} \cdot \cos\frac{\alpha-\beta}{2}$$

$$\text{sen}\,\alpha - \text{sen}\,\beta = 2 \cdot \cos\frac{\alpha+\beta}{2} \cdot \text{sen}\,\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2 \cdot \cos\frac{\alpha+\beta}{2} \cdot \cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2 \cdot \text{sen}\,\frac{\alpha+\beta}{2} \cdot \text{sen}\,\frac{\alpha-\beta}{2}$$

$$\tan\alpha \pm \tan\beta = \frac{\text{sen}(\alpha \pm \beta)}{\cos\alpha \cdot \cos\beta}$$

$$\cot\alpha \pm \cot\beta = \frac{\text{sen}(\beta \pm \alpha)}{\text{sen}\,\alpha \cdot \text{sen}\,\beta}$$

$$\text{sen}\,\alpha \cdot \cos\beta = \frac{1}{2}\,\text{sen}(\alpha+\beta) + \frac{1}{2}\,\text{sen}(\alpha-\beta)$$

$$\cos\alpha \cdot \cos\beta = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)$$

$$\text{sen}\,\alpha \cdot \text{sen}\,\beta = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$$

$$\tan\alpha \cdot \tan\beta = \frac{\tan\alpha + \tan\beta}{\cot\alpha + \cot\beta} = -\frac{\tan\alpha - \tan\beta}{\cot\alpha - \cot\beta}$$

$$\cot\alpha \cdot \cot\beta = \frac{\cot\alpha + \cot\beta}{\tan\alpha + \tan\beta} = -\frac{\cot\alpha - \cot\beta}{\tan\alpha - \tan\beta}$$

$$\cot\alpha \cdot \tan\beta = \frac{\cot\alpha + \tan\beta}{\tan\alpha + \cot\beta} = -\frac{\cot\alpha - \tan\beta}{\tan\alpha - \cot\beta}$$

Funções Circulares

E 5

Relações goniométricas

Relações

entre ângulos simples, duplos e meios ângulos

$\sin\alpha =$	$\cos\alpha =$	$\tan\alpha =$	$\cot\alpha =$
$\cos(90^\circ-\alpha)$	$\sin(90^\circ-\alpha)$	$\cot(90^\circ-\alpha)$	$\tan(90^\circ-\alpha)$
$\sqrt{1-\cos^2\alpha}$	$\sqrt{1-\sin^2\alpha}$	$\frac{1}{\cot\alpha}$	$\frac{1}{\tan\alpha}$
$2\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}$	$\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}$	$\frac{\sin\alpha}{\cos\alpha}$	$\frac{\cos\alpha}{\sin\alpha}$
$\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}$	$\frac{\cot\alpha}{\sqrt{1+\cot^2\alpha}}$	$\frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}}$	$\frac{\cos\alpha}{\sqrt{1-\cos^2\alpha}}$
$\sqrt{\cos^2\alpha-\cos(2\alpha)}$	$1-2\sin^2\frac{\alpha}{2}$	$\sqrt{\frac{1}{\cos^2\alpha}-1}$	$\sqrt{\frac{1}{\sin^2\alpha}-1}$
$\sqrt{\frac{1-\cos(2\alpha)}{2}}$	$\sqrt{\frac{1+\cos(2\alpha)}{2}}$	$\frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha}$	$\frac{\sqrt{1-\sin^2\alpha}}{\sin\alpha}$
$\frac{1}{\sqrt{1+\cot^2\alpha}}$	$\frac{1}{\sqrt{1+\tan^2\alpha}}$		
$\frac{2\cdot\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$	$\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$	$\frac{2\cdot\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$	$\frac{\cot^2\frac{\alpha}{2}-1}{2\cdot\cot\frac{\alpha}{2}}$

$\sin(2\alpha) =$	$\cos(2\alpha) =$	$\tan(2\alpha) =$	$\cot(2\alpha) =$
$2\cdot\sin\alpha\cdot\cos\alpha$	$\cos^2\alpha-\sin^2\alpha$	$\frac{2\cdot\tan\alpha}{1-\tan^2\alpha}$	$\frac{\cot^2\alpha-1}{2\cdot\cot\alpha}$
	$2\cos^2\alpha-1$	$\frac{2}{\cot\alpha-\tan\alpha}$	$\frac{1}{2}\cot\alpha-\frac{1}{2}\tan\alpha$
	$1-2\sin^2\alpha$		

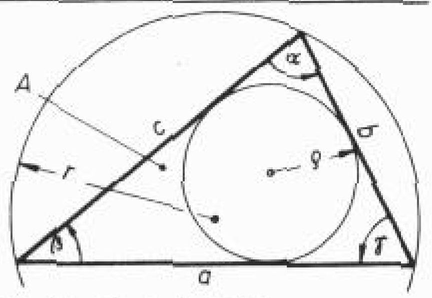
$\sin\frac{\alpha}{2} =$	$\cos\frac{\alpha}{2} =$	$\tan\frac{\alpha}{2} =$	$\cot\frac{\alpha}{2} =$
		$\frac{\sin\alpha}{1+\cos\alpha}$	$\frac{\sin\alpha}{1-\cos\alpha}$
$\sqrt{\frac{1-\cos\alpha}{2}}$	$\sqrt{\frac{1+\cos\alpha}{2}}$	$\frac{1-\cos\alpha}{\sin\alpha}$	$\frac{1+\cos\alpha}{\sin\alpha}$
		$\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$	$\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$

Funções circulares

Triângulo qualquer

E6

Triângulo qualquer



Teorema do seno

$$\text{sen}\,\alpha : \text{sen}\,\beta : \text{sen}\,\gamma = a : b : c$$

$$a = \frac{b}{\text{sen}\,\beta}\,\text{sen}\,\alpha = \frac{c}{\text{sen}\,\gamma}\,\text{sen}\,\alpha$$

$$b = \frac{a}{\text{sen}\,\alpha}\,\text{sen}\,\beta = \frac{c}{\text{sen}\,\gamma}\,\text{sen}\,\beta$$

$$c = \frac{a}{\text{sen}\,\alpha}\,\text{sen}\,\gamma = \frac{b}{\text{sen}\,\beta}\,\text{sen}\,\gamma$$

Teorema do co-seno

$$a^2 = b^2 + c^2 - 2\cdot bc\cdot\cos\alpha$$

$$b^2 = c^2 + a^2 - 2\cdot ac\cdot\cos\beta$$

$$c^2 = a^2 + b^2 - 2\cdot ab\cdot\cos\gamma$$

(O co-seno é negativo se o ângulo é obtuso)

Teorema das tangentes

$$\frac{a+b}{a-b} = \frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}} \quad\Bigg|\quad \frac{a+c}{a-c} = \frac{\tan\frac{\alpha+\gamma}{2}}{\tan\frac{\alpha-\gamma}{2}} \quad\Bigg|\quad \frac{b+c}{b-c} = \frac{\tan\frac{\beta+\gamma}{2}}{\tan\frac{\beta-\gamma}{2}}$$

Teorema da bissetriz

$$\tan\frac{\alpha}{2} = \frac{\varrho}{s-a} \quad\Big|\quad \tan\frac{\beta}{2} = \frac{\varrho}{s-b} \quad\Big|\quad \tan\frac{\gamma}{2} = \frac{\varrho}{s-c}$$

Área, raio do círculo inscrito e circunscrito

$$A = \frac{1}{2}\,b\,c\,\text{sen}\,\alpha = \frac{1}{2}\,a\,c\,\text{sen}\,\beta = \frac{1}{2}\,a\,b\,\text{sen}\,\gamma$$

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \varrho\,s$$

$$\varrho = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$r = \frac{1}{2}\cdot\frac{a}{\text{sen}\,\alpha} = \frac{1}{2}\cdot\frac{b}{\text{sen}\,\beta} = \frac{1}{2}\cdot\frac{c}{\text{sen}\,\gamma}$$

$$s = \frac{a+b+c}{2}$$

Funções circulares

Funções inversas

E7

Funções ciclométricas

Definição

	Função $y =$ arcsen x	arccos x	arctan x	arccot x
Função inversa de	$x = \text{sen}\, y$	$x = \cos y$	$x = \tan y$	$x = \cot y$
Domínio da definição	$-1 \leqq x \leqq +1$	$-1 \leqq x \leqq +1$	$-\infty < x < +\infty$	$-\infty < x < +\infty$
Valor principal no domínio	$-\frac{\pi}{2} \leqq y \leqq +\frac{\pi}{2}$	$\pi \geqq y \geqq 0$	$-\frac{\pi}{2} < y < +\frac{\pi}{2}$	$\pi > y > 0$

Relações fundamentais

$$\arccos x = \frac{\pi}{2} - \text{arcsen}\, x \qquad \text{arccot}\, x = \frac{\pi}{2} - \arctan x$$

Relações entre as funções arco

Se x é *positivo:*

$\text{arcsen}\, x =$	$\arccos x =$	$\arctan x =$	$\text{arccot}\, x =$
$\arccos \sqrt{1-x^2}$	$\text{arcsen} \sqrt{1-x^2}$	$\text{arcsen} \frac{x}{\sqrt{1+x^2}}$	$\text{arcsen} \frac{1}{\sqrt{1+x^2}}$
$\arctan \frac{x}{\sqrt{1-x^2}}$	$\arctan \frac{\sqrt{1-x^2}}{x}$	$\arccos \frac{1}{\sqrt{1+x^2}}$	$\arccos \frac{x}{\sqrt{1+x^2}}$
$\text{arccot} \frac{\sqrt{1-x^2}}{x}$	$\text{arccot} \frac{x}{\sqrt{1-x^2}}$	$\text{arccot} \frac{1}{x}$	$\arctan \frac{1}{x}$

Se x é negativo:

$$\text{arcsen}(-x) = -\text{arcsen}\, x \qquad \arccos(-x) = \pi - \arccos x$$

$$\arctan(-x) = -\arctan x \qquad \text{arccot}(-x) = \pi - \text{arccot}\, x$$

Teorema da adição

$$\text{arcsen}\, a \pm \text{arcsen}\, b = \text{arcsen}\left(a\sqrt{1-b^2} \pm b\sqrt{1-a^2}\right)$$

$$\arccos a \pm \arccos b = \arccos\left(a\, b \mp \sqrt{1-a^2} \cdot \sqrt{1-b^2}\right)$$

$$\arctan a \pm \arctan b = \arctan \frac{a \pm b}{1 \mp a\, b}$$

$$\text{arccot}\, a \pm \text{arccot}\, b = \text{arccot} \frac{a\, b \mp 1}{b \pm a}$$

Referência Bibliográfica

GIECK, Kurt. ***Manual de fórmulas técnicas***. São Paulo: Hemus Livraria Editora Ltda. 2 ed. 1979